SOLUTIONS TO EXAM 3, MATH 10560

1. *Calculate*

$$\lim_{n\to\infty} \frac{(\ln n)^2}{n}.$$

Solution: Use L'hospital rule:

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x} = \lim_{x\to\infty} \frac{((\ln x)^2)'}{x'} = \lim_{n\to\infty} \frac{2\frac{1}{x}\ln x}{1} = \lim_{x\to\infty} \frac{(2\ln x)'}{x'} = \lim_{x\to\infty} \frac{\frac{2}{x}}{1} = 0.$$

2. *Find* $\sum_{n=1}^{\infty} \frac{2^{2n}}{3 \cdot 5^{n-1}}$.

Solution:

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{3 \cdot 5^{n-1}} = \sum_{n=1}^{\infty} \frac{4^n}{3 \cdot 5^{n-1}} = \sum_{n=1}^{\infty} \frac{4 \cdot 4^{n-1}}{3 \cdot 5^{n-1}} = \frac{4}{3}\sum_{n=1}^{\infty} (\frac{4}{5})^{n-1} = \frac{4}{3}\frac{1}{1-\frac{4}{5}} = \frac{20}{3}.$$

3. *Discuss the convergence of the series*

$$\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}.$$

Solution: It converges conditionally. It's an alternating series and $b_n = \frac{1}{\sqrt{n}}$. Since b_n is decreasing and the limit of b_n is zero, the alternating series converges by the Alternating Series Test. But the series $\sum_{n=2}^{\infty} |\frac{(-1)^{n+1}}{\sqrt{n}}| = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ diverges since it's a p series and $p = \frac{1}{2} < 1$.

4. *Use Comparison Tests to determine which* **one** *of the following series is divergent.*

Solution: (a) $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}+1}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$, a p-series with $p = \frac{3}{2} > 1$.

(b) $\sum_{n=1}^{\infty} \frac{1}{n^2+8}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, a p-series with $p = 2 > 1$.

(c) $\sum_{n=1}^{\infty} \frac{n^2-1}{n^3+100}$ diverges by limit comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$, a p-series with $p = 1$.

(d) $\sum_{n=1}^{\infty} 7\left(\frac{5}{6}\right)^n$ converges since it is a geometric series with $r = \frac{5}{6} < 1$.

(e) $\sum_{n=1}^{\infty} \frac{n}{n+1}(\frac{1}{2})^n$ converges by comparison with $\sum_{n=1}^{\infty}(\frac{1}{2})^n$, a geometric series with $r = \frac{1}{2} < 1$.

5. *Which series below is the MacLaurin series (Taylor series centered at* 0*) for* $\frac{x^2}{1+x}$ *?*
Solution:

$$\frac{x^2}{1+x} = \frac{x^2}{1-(-x)} = x^2 \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^{n+2}.$$

6. *Find the degree 3 MacLaurin polynomial (Taylor polynomial centered at* 0*) for the function*

$$\frac{e^x}{1-x^2}$$

Solution: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}$. Thus

$$\frac{e^x}{1-x^2} = e^x \cdot \frac{1}{1-x^2} = (1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots)(1 + x^2 + \ldots) = 1 + x + \frac{3}{2}x^2 + \frac{7}{6}x^3 + \cdots.$$

7. *Which series below is a power series for* $\cos(\sqrt{x})$ *?*

Solution: Since $\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$, we have

$$\cos(\sqrt{x}) = \sum_{n=0}^{\infty}(-1)^n \frac{(\sqrt{n})^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n \frac{x^n}{(2n)!}.$$

8. *Calculate*

$$\lim_{x\to 0} \frac{\sin(x^3) - x^3}{x^9}.$$

Solution: Since $\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$, we have

$$\sin(x^3) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+3}}{(2n+1)!},$$

and

$$\lim_{x\to 0} \frac{\sin(x^3) - x^3}{x^9} = \lim_{x\to 0} \frac{x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots - x^3}{x^9} = \lim_{x\to 0} \frac{-\frac{x^9}{3!} + \cdots}{x^9} = -\frac{1}{6}.$$

9. *Does the series*

$$\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{2n}}$$

converge or diverge? Show your reasoning and state clearly any theorems or tests you are using.

Solution: Let $a_n = \frac{(n!)^n}{n^{2n}} = (\frac{n!}{n^2})^n$. Since

$$\lim_{n\to\infty} \frac{n!}{n^2} = \lim_{n\to\infty} \frac{n-1}{n} \cdot (n-2)! = \infty,$$

we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{n!}{n^2}\right)^n = \infty.$$

Hence $\lim_{n\to\infty} a_n \neq 0$. By the Test for Divergence, the series is divergent.

10. *Use the Integral Test to discuss whether the series* $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n}$ *converges.*

Solution: Let $f(x) = \frac{(\ln x)^2}{x}$. It is continuous and positive when $x > 1$. Note

$$f'(x) = \frac{2\ln x - (\ln x)^2}{x^2} = \frac{\ln x}{x^2}(2 - \ln x).$$

Then $f'(x) < 0$ and hence $f(x)$ is decreasing for $x > e^2$. Therefore, we can use the Integral Test. Next,

$$\int_1^{\infty} \frac{(\ln x)^2}{x} dx = \lim_{t\to\infty} \int_1^t (\ln x)^2 d(\ln x) = \lim_{t\to\infty} \frac{(\ln x)^3}{3}|_1^t = \lim_{t\to\infty} \frac{(\ln t)^3}{3} = \infty.$$

Hence the improper integral $\int_1^{\infty} \frac{(\ln x)^2}{x} dx$ is divergent. By the Integral Test, the series $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n}$ diverges.

11. *Find the radius of convergence and interval of convergence of the power series*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} (x-3)^n$$

Solution: Set $a_n = \frac{(-1)^n}{\sqrt{n}}(x-3)^n$. Using the Ratio Text,

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}}|x-3| = |x-3| < 1.$$

Hence, the radius of convergence is 1, and from $|x-3| < 1$ we get $2 < x < 4$. For the end points, when $x = 2$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is divergent since it is a p-series with $p = \frac{1}{2} < 1$; when $x = 4$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n (1)^n}{\sqrt{n}}$ which is convergent since it's an alternating series, and $b_n = \frac{1}{\sqrt{n}}$ are decreasing and $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$. Hence, the interval of convergence is $2 < x \leq 4$.

12. *(a) Show that*

$$\sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2}$$

provided that $|x| < 1$.

(b) Find

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(\sqrt{3})^{2n+1}}.$$

Solution: (a) Since $|x| < 1$, we have $|x^2| < 1$. Hence

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

(b) Integrate both the left and right hands of (a) to get

$$\int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx = \int \frac{1}{1+x^2} dx$$

$$\Rightarrow \sum_{n=0}^{\infty} \int (-1)^n x^{2n} dx = \int \frac{1}{1+x^2} dx$$

$$\Rightarrow \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \arctan x + C.$$

Letting $x = 0$, we have $C = 0$. Hence, we have

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \arctan x.$$

Let $x = \frac{1}{\sqrt{3}}$. We get

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(\sqrt{3})^{2n+1}} = \arctan \frac{1}{\sqrt{3}} = \frac{\pi}{6}.$$